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COUPLED FIXED POINT THEOREM IN PARTIALLY ORDERED COMPLETE ULTRA METRIC SPACE

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Abstract: In this paper coupled fixed point theorem in partially ordered complete ultra metric space is proved using fixed monotone property.
Keywords: Coupled fixed point, Ultra metric space, Mixed monotone property.

1. INTRODUCTION

Fixed point theory is a very useful tool in solving variety of problems in the control theory, economic theory, nonlinear analysis and, global analysis. the banach contraction principle is the most famous, most simplest, and one of the most versatile elementary results in the fixed point theory. A huge amount of literature is witnessed on applications, generalizations, and extensions of this principle carried out by several authors in different directions, for example, by weakening the hypothesis, using different setups, and considering different mappings.

Fixed point theorems in partially ordered metric spaces and applications are investigated in [1]. Contractive mapping theorems in partially ordered sets and application to ordinary differential equation are investigated in [2]. A fixed point theorem in partially ordered sets and some applications to matrix equations is investigated in [3]. A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi metric spaces is investigated in [4]. Dislocated Topologies are investigated in [5].

2. BASIC CONCEPTS

Definition 1.1: Let (X, ≤) be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have mixed monotone property if F is non decreasing monotone in its first argument and is non increasing monotone in its second argument, that is for any $x, y \in X$,

$x_1, x_2 \in X, x_1 \le x_2, F(x_1, y) \le F(x_2, y),$

$y_1, y_2 \in X, y_1 \le y_2, F(x, y_1) \ge F(x, y_2).$

Definition 1.2: An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x, F(y, x) = y.$

Definition 1.3: Let X be a non empty set and let $d : X \times X \to [0,\infty)$ be a function, called a distance function we need the following conditions:

i.) $d(x, x) \ge 0$

ii.) $d(x, y) = d(y, x) = 0,$ then $x = y$

iii.) $d(x, y) = d(y, x)$

iv.) $d(x, y) \le d(x, z) + d(z, y)$

v.) $d(x, y) \le \max\{d(x, z), d(z, y)\} \forall x, y, z \in X.$

If d satisfies condition (i) – (iv), then it is called metric on X. If a metric d satisfies the strong triangle inequality (v), then it is called an ultra metric space.

Definition 1.4: A sequence $\{x_n\}$ ultra metric converges to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$

Definition 1.5: A sequence $\{x_n\}$ in an ultra metric space (X, d) is called Cauchy if $\forall \varepsilon > 0, n_0 \in N$ such that $\forall m, n \ge n_0, d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon.$

Definition 1.6: An ultra metric space (X, d) is called complete if every Cauchy sequence in it is convergent.

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3. MAIN RESULT

Theorem 2.1: Let $(X,\leq)$ be a partially ordered set and (X,d) is a complete ultra metric space. Let $F:X\times X\to X$ be a continuous mapping having the fixed monotone property on X such that there exist two elements $x_0,y_0\in X$ with $x_0\leq F(x_0,y_0)$ and $y_0\geq F(y_0,x_0)$ suppose that there exist a non negative real number $k\in[0,\frac{1}{2})$ such that $(a)d(F(x,y),F(u,v))\leq\frac{k}{2}[d(x,u)+d(y,v)]$ $\forall x,y,u,v\in X$ with $x\geq u$ and $y\leq v$. Then F has coupled fixed point in X.

Proof: Let $x_0,y_0\in X$ with $x_0\leq F(x_0,y_0)$ and $y_0\geq F(y_0,x_0)-(1)$.

Define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1}=F(x_n,y_n)$ and $y_{n+1}=F(x_n,y_n)-(2)$ $\forall n=0,1,2...$

We claim that $\{x_n\}$ is monotonic increasing andd $\{y_n\}$ is monotonic decreasing i.e. $x_n\leq x_{n+1}$ and $y_n\geq y_{n+1}$ $\forall n=0,1,2...$

So, from (1) and (2) we have,

$x_0\leq F(x_0,y_0)$, $y_0\geq F(y_0,x_0)$ and

$x_1=F(x_0,y_0)$, $y_1=F(y_0,x_0)$

Thus $x_0\leq x_1$ and $y_0\geq y_1$.

Thus result is true for n = 0.

Now, suppose that the result holds for some n i.e.

$x_n\leq x_{n+1}$ and $y_n\geq y_{n+1}$.

We shall prove that the result is true for n+1.

Now, by assumed hypothesis

$x_n\leq x_{n+1}$ and $y_n\geq y_{n+1}$ then by mixed monotone property of F, we have

$x_{n+2}=F(x_{n+1},y_{n+1})\geq F(x_n,y_{n+1})\geq F(x_n,y_n)=x_{n+1}$

and $\mathrm{y}_{n+2}=F(y_{n+1},x_{n+1})\leq F(y_n,x_{n+1})\leq F(y_n,x_n)=y_{n+1}$

Thus by mathematical induction result hold for all n in N.

So, $x_0\leq x_1\leq x_2\leq x_3\leq...\leq x_n\leq x_{n+1}\leq...$

and $\mathrm{y}_0\geq y_1\geq y_2\geq y_3\geq...\geq y_n\geq y_{n+1}\geq...$

Since $x_{n-1}\leq x_n$ and $y_{n-1}\geq y_n$, from (a) we have,

$$d(F(x_n,y_n),F(x_{n-1},y_{n-1}))\leq\frac{k}{2}[d(x_n,x_{n-1})+d(y_n,y_{n-1})]$$

Thus $d(x_{n+1},x_n)\leq\frac{k}{2}[d(x_n,x_{n-1})+d(y_n,y_{n-1})]--(3)$

Similarly $\mathrm{y}_{n-1}\geq y_n$ and $x_{n-1}\leq x_n$, from (a) we have,

$$d(F(y_{n-1},x_{n-1}),F(y_n,x_n))\leq\frac{k}{2}[d(y_{n-1},y_n)+d(x_{n-1},x_n)]$$

Thus $d(y_n,y_{n+1})\leq\frac{k}{2}[d(y_{n-1},y_n)+d(x_{n-1},x_n)]--(4)$

Adding (3) and (4) we get,

$d(x_{n+1},x_n)+d(y_n,y_{n+1})\leq k[d(y_n,y_{n-1})+d(x_n,x_{n-1})]$

Let $d_n=d(x_{n+1},x_n)+d(y_n,y_{n+1})$

So, $d_n\leq kd_{n-1}$ Similarly $d_{n-1}\leq kd_{n-2}$

Thus, $d_n\leq kd_{n-1}\leq \mathrm{k}^2d_{n-2}\leq...\leq k^n d_0$.

$$\Rightarrow \lim_{n\to\infty} d_n=\lim_{n\to\infty}[d(x_{n+1},x_n)+d(y_{n+1},\mathrm{y}_n)]=0$$

Thus $\lim_{n\to\infty} d(x_{n+1},x_n)=\lim_{n\to\infty} d(y_{n+1},\mathrm{y}_n)=0.$

for each $m \geq n$ we have ,

$$d(x_m, x_n) \leq \max[d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), ..., d(x_{m-1}, x_m)]$$

$$\leq \max[\frac{k}{2}[d(x_n, x_{n-1}) + d(y_n, y_{n-1})], \frac{k}{2}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)], \frac{k}{2}[d(x_{n+2}, x_{n+1}) + d(y_{n+2}, y_{n+1})], ...,$$

$$\frac{k}{2}[d(x_{m-1}, x_{m-2}) + d(y_{m-1}, y_{m-2})]$$

$$\leq \max[\frac{k}{2} d_{n-1}, \frac{k^2}{2} d_{n-1}, \frac{k^3}{2} d_{n-1}, ..., \frac{k^{n+m}}{2} d_{n-1}]$$

$$\leq \frac{k}{2} d_{n-1} --- (5)$$

$$d(y_m, y_n) \leq \max[d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), ..., d(y_{m-1}, y_m)]$$

$$\leq \max[\frac{k}{2}[d(y_{n-1}, y_n) + d(x_{n-1}, x_n)], \frac{k}{2}[d(y_n, y_{n+1}) + d(x_n, x_{n+1})], ..., \frac{k}{2}[d(x_{m-1}, x_{m-2}) + d(y_{m-1}, y_{m-2})]$$

$$\leq \max[\frac{k}{2} d_{n-1}, \frac{k^2}{2} d_{n-1}, \frac{k^3}{2} d_{n-1}, ..., \frac{k^{n+m}}{2} d_{n-1}]$$

$$\leq \frac{k}{2} d_{n-1} --- (6)$$

Adding (5) and (6) we get,

$$d(x_m, x_n) + d(y_m, y_n) \leq k d_{n-1}$$

$$\leq k^n d_0$$

$$\to 0 \text{ as } n \to \infty \text{ as } k < 1/2$$

therefore $\{x_n\}$ and $\{y_n\}$ are cauchy sequences in X.

Since X is complete ultra metric space, $\exists x, y \in X$ such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$

Thus by taking limit as $n \to \infty$ in (2) we get,

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}) = F(\lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}) = F(x, y).$$

$$y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}) = F(\lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}) = F(y, x).$$

$\therefore x = F(x, y)$ and

$y = F(y, x)$.

Thus F has a coupled fixed point.

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